The Lagrangian Formulation of the Simple Pendulum

Omar Corona Tejeda

omarct1989@ciencias.unam.mx

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Abstract

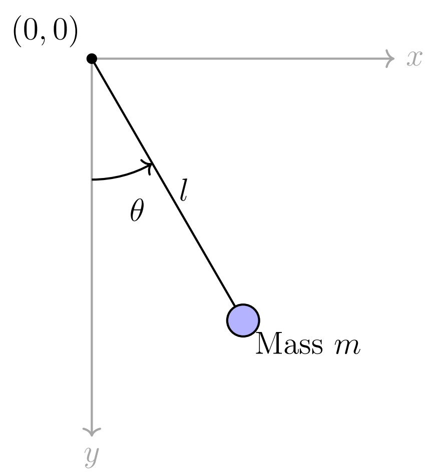
We present a concise derivation of the Lagrangian and equation of motion for a simple pendulum of mass m and length l. Special attention is given to the sign conventions associated with the gravitational constant g and to the equivalence between the potential energies $V = -mgl\cos\theta$ and $V = mgl(1-\cos\theta)$.

1 Introduction

The simple pendulum is one of the most fundamental dynamical systems in classical mechanics. Using the angle θ measured from the vertical, we derive the Lagrangian, obtain the Euler–Lagrange equation, and clarify certain issues about sign conventions and potential-energy choices.

2 Geometry of the Pendulum

Consider a point mass m attached to a rigid massless rod of length l. Using the pivot as the origin and measuring the angle θ from the vertical downward direction, the Cartesian coordinates of the mass are



Simple pendulum of length l and mass m displaced by an angle θ.

$$x = l\sin\theta, \qquad y = -l\cos\theta. \tag{1}$$

The velocities are

$$\dot{x} = l\cos\theta\,\dot{\theta}, \qquad \dot{y} = l\sin\theta\,\dot{\theta}. \tag{2}$$

3 Kinetic and Potential Energies

The kinetic energy is

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}ml^2\dot{\theta}^2. \tag{3}$$

The gravitational potential is

$$V = mgy = -mgl\cos\theta. \tag{4}$$

This expression is obtained directly from $y = -l\cos\theta$ with the convention that $g > 0$ and upward is positive.

Alternatively, one may add a constant mgl to shift the zero of potential so that $V(0) = 0$:

$$V(\theta) = mgl(1 - \cos\theta). \tag{5}$$

Both potentials differ only by a constant and yield identical equations of motion.

4 The Lagrangian

Using $L = T - V$, the Lagrangian becomes

$$L(\theta, \dot{\theta}) = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta. \tag{6}$$

5 Equation of Motion

The Euler–Lagrange equation for the generalized coordinate θ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0. \tag{7}$$

Since

$$\frac{\partial L}{\partial\dot{\theta}} = ml^2\dot{\theta}, \qquad \frac{\partial L}{\partial\theta} = -mgl\sin\theta, \tag{8}$$

the equation of motion becomes

$$ml^2\ddot{\theta} + mgl\sin\theta = 0, \tag{9}$$

or, after dividing by ml^2,

$$\boxed{\ddot{\theta} + \frac{g}{l}\sin\theta = 0.} \tag{10}$$

6 Small-Angle Approximation

For small oscillations, $\sin\theta \approx \theta$, giving

$$\boxed{\ddot{\theta} + \frac{g}{l}\theta = 0,} \tag{11}$$

a simple harmonic oscillator with angular frequency

$$\omega = \sqrt{\frac{g}{l}}. \tag{12}$$

7 Sign Conventions and Potential Choice

A common source of confusion arises from sign conventions in the potential. It is important to emphasize:

- The gravitational constant g is always positive.
- The sign of the potential comes from the coordinate convention.
- The potentials $-mgl\cos\theta$ and $mgl(1-\cos\theta)$ differ only by a constant and lead to the same dynamics.

8 Solution of the Pendulum Equation

The motion of a simple pendulum of length l and mass m is governed by the nonlinear differential equation

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0. \tag{13}$$

Energy Integral

Equation (13) can be integrated once by multiplying both sides by $\dot{\theta}$:

$$\dot{\theta}\,\ddot{\theta} + \frac{g}{l}\,\dot{\theta}\sin\theta = 0.$$

This gives the conserved energy

$$\frac{1}{2}\dot{\theta}^2 - \frac{g}{l}\cos\theta = C, \tag{14}$$

where C is determined by the initial conditions.

Solving (14) for $\dot{\theta}$ we obtain the first–order equation

$$\dot{\theta} = \sqrt{2\left(C + \frac{g}{l}\cos\theta\right)}. \tag{15}$$

Exact Nonlinear Solution

From the energy form we may write

$$\frac{d\theta}{\sqrt{2\left(C + \frac{g}{l}\cos\theta\right)}} = dt.$$

Using the turning point condition $\dot{\theta} = 0$ at $\theta = \theta_0$, the constant becomes

$$C = -\frac{g}{l}\cos\theta_0.$$

Thus,

$$\frac{d\theta}{\sqrt{\frac{2g}{l}\left(\cos\theta - \cos\theta_0\right)}} = dt.$$

The exact solution is expressed in terms of the Jacobi elliptic function[1] :

$$\theta(t) = 2\arcsin\left(k\,\mathrm{sn}\left(\sqrt{\frac{g}{l}}\,t,\ k\right)\right), \qquad k = \sin\frac{\theta_0}{2}. \tag{16}$$

Small–Angle Approximation

If θ is small, $\sin\theta \approx \theta$, and (13) becomes the linear equation

$$\ddot{\theta} + \omega^2\theta = 0, \qquad \omega = \sqrt{\frac{g}{l}}.$$

Its general solution is

$$\theta(t) = A\cos(\omega t) + B\sin(\omega t), \tag{17}$$

or, using amplitude and phase,

$$\theta(t) = \theta_0\cos(\omega t + \phi). \tag{18}$$

[1]The Jacobi elliptic function $\mathrm{sn}(u,k)$ is defined as the inverse of the incomplete elliptic integral of the first kind. If $\phi = \mathrm{am}(u,k)$ is the Jacobi amplitude, then

$$u = F(\phi,k) = \int_0^{\phi}\frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}},$$

and the function is defined by

$$\mathrm{sn}(u,k) = \sin\big(\mathrm{am}(u,k)\big).$$

It satisfies the nonlinear differential equation

$$\frac{d}{du}\mathrm{sn}(u,k) = \mathrm{cn}(u,k)\,\mathrm{dn}(u,k), \qquad \left(\frac{d}{du}\mathrm{sn}(u,k)\right)^2 = (1-\mathrm{sn}^2(u,k))(1-k^2\mathrm{sn}^2(u,k)).$$

The function is periodic in u with real period $4K(k)$, where

$$K(k) = \int_0^{\pi/2}\frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}}$$

is the complete elliptic integral of the first kind. For $k \to 0$ it reduces to the sine function, $\mathrm{sn}(u,0) = \sin u$, and for $k \to 1$ it approaches the hyperbolic tangent, $\mathrm{sn}(u,1) = \tanh u$.

The period in this approximation is

$$T_0 = 2\pi\sqrt{\frac{l}{g}}.$$

9 Conclusion

We derived the Lagrangian, clarified the role of sign conventions, and showed the equivalence of common potential-energy expressions. The resulting equation of motion is

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0,$$

the standard equation for a simple pendulum.